

The Pompeiu problem

A. G. Ramm

Abstract

Let $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'$, where \mathcal{S}' is the Schwartz class of distributions, and

$$\int_{\sigma(D)} f(x) dx = 0 \quad \forall \sigma \in G, \quad (*)$$

where $D \subset \mathbb{R}^n$ is a bounded domain, the closure \bar{D} of which is diffeomorphic to a closed ball. Then the complement of \bar{D} is connected and path connected. Here G denotes the group of all rigid motions in \mathbb{R}^n . This group consists of all translations and rotations.

It is conjectured that if $f \neq 0$ and $(*)$ holds, then D is a ball. Two other conjectures, equivalent to the above one, are formulated and discussed.

MSC: 35J05, 31B20

Key words: Symmetry problems; Pompeiu problem.

1 Introduction

In this paper the problem known as the Pompeiu problem is formulated and discussed. This problem originated in Pompeiu's paper [7], of 1929. The problem in a modern formulation is stated below as Conjecture 1, and is still open.

Dimitrie Pompeiu (1873-1954) was born in Romania and got his Ph.D in 1905 at the Sorbonne, in Paris, under the direction of H. Poincaré. He is known mainly for the Pompeiu problem and for the Cauchy-Pompeiu formula in complex analysis.

Let us formulate the Pompeiu problem as it is understood today.

Let $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'$, where \mathcal{S}' is the Schwartz class of distributions, and

$$(1) \quad \int_{\sigma(D)} f(x) dx = 0 \quad \forall \sigma \in G,$$

where G is the group of all rigid motions of \mathbb{R}^n , consisting of all translations and rotations, and $D \subset \mathbb{R}^n$ is a bounded domain, the closure \bar{D} of which is diffeomorphic to a closed ball.

Under these assumptions the complement of \bar{D} in \mathbb{R}^n is connected and path connected by the isotopy extension theorem, see [4].

The distribution space \mathcal{S}' in the assumption $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'$ can be replaced without essential changes in the argument by the distribution space \mathcal{D}' , where \mathcal{D} is the space of $C_0^\infty(\mathbb{R}^3)$ functions.

In [7] the following question was raised:

Does (1) imply that $f = 0$?

If yes, then we say that D has *P-property* (Pompeiu's property), and write $D \in P$. Otherwise, we say that D *fails to have P-property*, and write $D \in \bar{P}$. Pompeiu claimed in 1929 that every plane bounded domain has *P-property*. This claim turned out to be false: a counterexample was given 15 years later in [2]. The counterexample is a domain D which is a disc, or a ball in \mathbb{R}^n for $n > 2$. If D is a ball, then there are $f \neq 0$ for which equation (1) holds. The set of all $f \neq 0$, for which equation (1) holds, was constructed in [8]. There are infinitely many (a continuum) such f . Let us give the counterexample mentioned above.

Example 1. *Suppose that $D \subset \mathbb{R}^n$ is a ball B centered at the origin and of radius a . Then the Fourier transform of its characteristic function χ is*

$$(2) \quad \tilde{\chi}(\xi) = \int_B e^{i\xi \cdot x} dx = (2\pi a)^{n/2} \frac{J_{n/2}(a|\xi|)}{|\xi|^{n/2}},$$

where $J_{n/2}$ is the Bessel function. It follows that if $|\xi| = s_{j,n}/a$, where $s_{j,n}$ is any positive zero of the Bessel function $J_{n/2}(s)$, then $\tilde{\chi}(\xi)$ has a spherical set of zeros. This implies, as follows from Theorem 3, proved below, that there are $f \neq 0$ for which relation (1) holds.

A bibliography on the Pompeiu problem (*P-problem*) can be found in [15] and in [8]. The current formulation of the *P-problem* is the following:

Prove that if $D \subset \mathbb{R}^n$ is a bounded domain diffeomorphic to a ball and $D \in \bar{P}$, then D is a ball.

We use the word ball also in the case $n = 2$, when this word means disc, and discuss the *P-problem* in detail. This problem leads to some problems of general mathematical interest: a symmetry problem for partial differential equations, see Conjecture 2 below, and a problem in harmonic analysis, see Conjecture 3 below.

Let us make the following standing assumptions:

Assumptions A:

$A_1)$: D is a bounded domain, the closure of which is diffeomorphic to a closed ball, the boundary S of D is a closed connected C^1 -smooth surface,

$A_2)$: D fails to have *P-property*.

Our first conjecture is:

Conjecture 1. *If Assumptions A hold, then D is a ball.*

In Section 2 this Conjecture is discussed. We prove that Conjecture 1 is equivalent to a symmetry problem for a partial differential equation. Namely, it is equivalent to the following conjecture.

Conjecture 2. *If problem (3) (see below) has a solution, then D is a ball.*

Several symmetry problems were studied by the method similar to the one used in the proof of Theorem 1, below, see also [11]-[13].

Conjectures 1 and 2 are equivalent to the following conjecture:

Conjecture 3. *If Assumption A_1 holds and the Fourier transform $\tilde{\chi}_D$ of the characteristic function χ_D of the domain D has a spherical surface of zeros, then D is a ball.*

2 Discussion of the Conjectures

It is proved in [14] that if Assumptions A hold, then the boundary S of D is real-analytic. It is proved in Theorem 3 below, that if Assumptions A hold, then the problem

$$(3) \quad (\nabla^2 + k^2)u = 1 \quad \text{in } D, \quad u|_S = 0, \quad u_N|_S = 0, \quad k^2 = \text{const} > 0,$$

has a solution. In (3) $N = N_s$ is the outer unit normal to S pointing out of D , $s \in S$ is a point on S .

Therefore, if (1) holds, then problem (3) has a solution.

Let us prove that if problem (3) has a solution, then $\tilde{\chi}_D$ has a spherical set of zeros, where χ_D is the characteristic function of the domain D . To prove this, let us multiply (3) by $e^{ik\alpha \cdot x}$, where $\alpha \in S^2$ is an arbitrary unit vector and S^2 is the unit sphere in \mathbb{R}^3 , and integrate with respect to x over D . Using an integration by parts and the boundary conditions (10) for u , one gets the desired relation:

$$(4) \quad \tilde{\chi}_D(k\alpha) = 0 \quad \forall \alpha \in S^2.$$

Thus, $\tilde{\chi}_D$ has a spherical set of zeros.

Conjectures 1, 2, and 3 are equivalent in the sense that each of them claims that D is a ball. But these three Conjectures are also equivalent in the sense that if one of them is correct, then so are the remaining two.

Indeed, in the proof of Theorem 3 equation (3) was derived from equation (1), and the relation

$$(5) \quad \tilde{\chi}_D(\xi) = 0, \quad |\xi| = \text{const} > 0,$$

was derived from equation (3). From the proof of Theorem 3 it follows that equation (5) implies the equation

$$(6) \quad \tilde{\chi}(\xi) = (\xi^2 - k^2)\tilde{u}(\xi),$$

(see equation (23) below, in the proof of Theorem 3), and the inverse Fourier transform of (6) yields equation (3) and the boundary conditions in (3). In this sense problem (3) is equivalent to relation (5), and Conjectures 2 and 3 are equivalent in the sense that if one of them is correct, so is the other one.

Furthermore, from the proof of Theorem 3 it follows that (5) implies (1). Indeed, equation (5) implies equation

$$(7) \quad \tilde{f}(\xi)\overline{\tilde{\chi}(g^{-1}\xi)} = 0,$$

(see equation (21) below), and the inverse Fourier transform of this equation yields relation (1).

In this sense relations (1), (3) and (5) are equivalent, and Conjectures 1, 2 and 3, are equivalent in the sense that if one of them is correct, so are the other two.

Problem (3) is an open symmetry problem of long standing for partial differential equations. Let us formulate another open symmetry problem for partial differential equations of long standing, known as M. Schiffer's conjecture:

Conjecture 4. *If the problem*

$$(8) \quad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \quad u_N|_S = \text{const} \neq 0, \quad u|_S = 0, \quad k^2 = \text{const} > 0,$$

has a solution, then D is a ball.

Note that Conjecture 2 can be formulated in the form similar to (8):

Conjecture 5. *If the problem*

$$(9) \quad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \quad u_N|_S = 0, \quad u|_S = \text{const} \neq 0, \quad k^2 = \text{const} > 0,$$

has a solution, then D is a ball.

Conjecture 5 is equivalent to Conjecture 2. Indeed, if (3) holds, then one can look for the solution u of the form $u = v + c$, where c is a constant. The boundary conditions in (3) imply $v = -c$ on S , and $v_N = 0$ on S . Let us choose $c = 1/k^2$. Then the differential equation (3) implies

$$(\nabla^2 + k^2)v = 1 - k^2c = 0.$$

Therefore, v solves problem (9). Conversely, if v solves problem (9), then $u = v + c$ solves (3) if c is a suitable constant. \square

Conjecture 5 is *not* equivalent to Conjecture 4.

The results, on which our discussion of Conjectures 1, 2 and 3 is based, are Theorems 1, 2 and 3. Theorems 1 and 2 were proved originally in [9], [8], and in the book [10], Chapter 11. A result, equivalent to Theorem 3, had been proved originally in the paper [1] by a considerably longer and more complicated argument. Our proofs are borrowed essentially from [10], Chapter 11. In the paper [6] the null-varieties of the Fourier transform of the characteristic function of a bounded domain D are studied. The properties of these varieties and the geometrical properties of D are related, of course, but it is not clear in what way they are related. Conjecture 3, if it is proved, is an interesting example of such a relation.

In Section 3 a relation of the Pompeiu problem in \mathbb{R}^2 to analyticity of f is discussed. It is proved that if the domain $D \in P$, $f \in L^1_{loc}(\mathbb{R}^2)$, and if $\int_{\partial\sigma(D)} f dz = 0 \quad \forall \sigma \in G$, then f is an entire function. An earlier discussion of this result can be found in [16], and a new short proof of a result from [16] is given.

In Section 4 a new approach to the Pompeiu problem is outlined and a new Conjecture is formulated.

To make our presentation essentially self-contained, proofs of Theorems 1, 2 and 3, are included in this paper.

Theorem 1. *If Assumptions A hold, then*

$$(10) \quad [s, N] = u_N, \quad \forall s \in S,$$

where $[s, N]$ is the cross product in \mathbb{R}^3 , and u is a vector-function that solves the problem

$$(11) \quad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \quad u|_S = 0.$$

If $n = 2$, then D is a plane domain, S is a curve, diffeomorphic to a circle, u is a scalar solution to equation (4), and equation (3) yields $s_1 N_2 - s_2 N_1 = u_N$, $\forall s \in S$, where N_j , $j = 1, 2$, are Cartesian coordinates of the unit normal N to S . Indeed, if $n = 2$ then the cross product of two vectors $[s_1 e_1 + s_2 e_2, N_1 e_1 + N_2 e_2]$ is calculated by the formula $[s, N] = (s_1 N_2 - s_2 N_1) e_3$, where e_3 is a unit vector, orthogonal to the plane domain D , and the triple $\{e_j\}_{j=1}^3$ is a standard orthonormal basis in \mathbb{R}^3 .

Let us state the following characterization of spheres.

Theorem 2. *If S is a smooth surface homeomorphic to a sphere and $[s, N] = 0$ on S , then S is a sphere.*

The proof of Theorem 2 will be given in the coordinate system in which the condition $[s, N] = 0$ on S is valid.

The following conclusion is an immediate consequence of Theorem 2:

The conclusion of Conjecture 1 will be established if one proves, under the Assumptions A, that $[s, N] = 0$ on S .

Let us start by proving Theorem 2, then Theorem 1 is proved, and, finally, we prove Theorem 3. It is assumed throughout, except in Section 3, that $n = 3$. Our proofs of Theorems 2 and 3 can be used for any integer $n \geq 2$ without any essential changes. The proof of Theorem 1 uses the notion of the cross-product, and by this reason its proof should be modified for $n > 3$.

Proof of Theorem 2. Let $n = 3$ and assume that $s = s(p, q)$ is a parametric equation of the surface S . The normal N to S is a vector directed along the vector $[s_p, s_q]$, where s_p denotes the partial derivative of the function $s(p, q)$ with respect to the parameter p . The assumption $[s, N] = 0$ on S , yields

$$(12) \quad [s, [s_p, s_q]] = s_p s \cdot s_q - s_q s \cdot s_p = 0,$$

where $s \cdot s_q$ is the dot product of two vectors in \mathbb{R}^3 . At a non-singular points of S , the vectors s_p and s_q are linearly independent. The surface S is smooth, so its points are non-singular. Therefore equation (12) implies $s \cdot s_q = 0$ and $s \cdot s_p = 0$, so

$$(13) \quad \frac{\partial s \cdot s}{\partial p} = 0, \quad \frac{\partial s \cdot s}{\partial q} = 0.$$

Therefore

$$(14) \quad s \cdot s = \text{const.}$$

This is an equation of a sphere in the coordinate system with the origin at the center of the sphere. Theorem 2 is proved. \square

Proof of Theorem 1. Let $n = 3$ and \mathcal{N} denote the set of all smooth solutions to (11) in a ball B , containing D . Multiply (3) by an arbitrary solution h to equation (11) in a ball B , containing D , integrate by parts, take into account the boundary conditions in (3), and get the relation

$$\int_D h(x) dx = 0 \quad \forall h \in \mathcal{N}.$$

Since $h \in \mathcal{N}$ implies $h(gx) \in \mathcal{N}$ for all g , where g is an arbitrary rotation in \mathbb{R}^3 about the origin O , one obtains

$$(15) \quad \int_D h(gx) dx = 0 \quad \forall h \in \mathcal{N}, \forall g.$$

Let $O \in D$, and take an arbitrary straight line ℓ passing through O and directed along a unit vector α . Let $g = g(\phi)$ be the rotation about ℓ by an angle ϕ counterclockwise.

Differentiate (15) with respect to ϕ and then set $\phi = 0$, see [9]. A similar argument has been used in [11]-[13]. The result is

$$(16) \quad \int_D \nabla h(x) \cdot [\alpha, x] dx = 0,$$

where $[\alpha, x]$ is the cross product and \cdot stands for the inner product in \mathbb{R}^3 . Equation (16) is invariant with respect to translations because $\int_D \nabla h(x) dx = 0 \forall h \in \mathcal{N}$. Indeed, $h \in \mathcal{N}$ implies $\nabla h \in \mathcal{N}$. Using the divergence theorem, the relation $\nabla \cdot [\alpha, x] = 0$, valid for any constant vector α , and the arbitrariness of α , one derives from (16) the following relation

$$(17) \quad \int_S h(s)[s, N] ds = 0, \quad \forall h \in \mathcal{N},$$

which is also invariant with respect to translations. Indeed, N is invariant under translations because $[s_p, s_q]$ is, and $\int_S h(s)[a, N] ds = 0$ for any constant vector a because $\int_S h(s)N ds = \int_D \nabla h dx = 0$, as was pointed out above. Let us derive from (17) equation (10). We need the following result.

Lemma 1. *The orthogonal complement of the set M of the restrictions of all $h \in \mathcal{N}$ to S is a finite-dimensional space spanned by the functions u_{jN} , where $\{u_j\}_{j=1}^J$ is the basis of the eigenspace of the Dirichlet Laplacian in D , corresponding to the eigenvalue k^2 .*

Remark 1. It follows from (17) that $[s, N]$ is orthogonal in $L^2(S)$ to the set M . Therefore, by Proposition 1, each of the three components of $[s, N]$ must be linear combinations of the functions u_{jN} , $1 \leq j \leq J$. In other words, equation (10) holds.

Proof of Lemma 1. Note that the result of Lemma 1 is equivalent to the assertion that the boundary value problem

$$(18) \quad (\nabla^2 + k^2)h = 0 \quad \text{in } D, \quad h|_S = f$$

is solvable if and only if

$$(19) \quad \int_S f u_{jN} ds = 0, \quad 1 \leq j \leq J.$$

where u_j , $1 \leq j \leq J$, is a basis of the solutions to problem (11).

The *necessity* of conditions (19) is proved by the relation

$$0 = \int_D u_j (\nabla^2 + k^2) h dx = - \int_S f u_{jN} ds,$$

where an integration by parts and the boundary condition $u_j = 0$ on S were used, and equation (11) for h was taken into account.

The *sufficiency* of conditions (19) is proved as follows. Denote by $H^m(D)$ the usual Sobolev spaces. Given an $f \in H^{3/2}(S)$, construct an arbitrary $F \in H^2(D)$, such that $F|_S = f|_S$, and define $h := w + F$, where

$$(20) \quad (\nabla^2 + k^2)w = -(\nabla^2 + k^2)F \quad \text{in } D, \quad w|_S = 0.$$

If such w exists, then $h = w + F$ solves problem (18). For the existence of w it is necessary and sufficient that

$$\int_D (\nabla^2 + k^2)F u_j dx = 0, \quad 1 \leq j \leq J.$$

An integration by parts shows that these conditions are equivalent to conditions (19) because u_j solve problem (11). Thus, Lemma 1 is proved. \square

Equation (17) says that $[s, N]$ is orthogonal to the set M , that is, to the restrictions of all $h \in \mathcal{N}$ to S .

Lemma 2. *The set M is dense in $L^2(S)$ in the set of all $\psi \in H^{3/2}(S)$ for which the boundary problem (20) is solvable.*

This Lemma, equation (17), and Lemma 1 imply (10).

Proof of Lemma 2. Assume the contrary. Then for some $f \in H^{3/2}(S)$, $f \neq 0$, problem (18) is solvable and

$$\eta(y) := \int_S f(s)\psi(s, y)ds = 0 \quad \forall y \in D' := \mathbb{R}^3 \setminus D,$$

where $\psi(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|} \in \mathcal{N}$ for $y \in B'$, that is, outside a ball containing D . The function η is a simple-layer potential which vanishes in B' , and by the unique continuation property for solutions of the homogeneous Helmholtz equation, $\eta = 0$ everywhere in D' . Thus, it vanishes on S . Therefore, η solves problem (11). By the jump relation for the normal derivative of η across S , one has $f = \eta_N$, where η_N is the limiting value of the normal derivative of η on S from inside D . If problem (17) is solvable, then, as we have proved, f is orthogonal to all functions u_{jN} . The function η_N is a linear combination of these functions. This and the relation $f = \eta_N$ prove that $f = 0$. Consequently, we have proved the claimed density of M in the set of all $H^{3/2}(S)$ -functions f for which problem (18) is solvable. Lemma 2 is proved. \square

This completes the proof of Theorem 2. \square

Theorem 3. *Suppose Assumptions A hold and relation (1) holds for some $f \neq 0$. Then problem (3) has a solution. Conversely, if problem (3) has a solution, then there exists $f \neq 0$ such that relation (1) holds.*

Proof of Theorem 3. Write (1) as

$$\int_{\mathbb{R}^3} f(gx + y)\chi(x)dx = 0 \quad \forall y \in \mathbb{R}^3, \quad \forall g \in G,$$

where $\chi(x)$ is the characteristic function of D . Applying the Fourier transform and the convolution theorem one gets

$$(21) \quad \tilde{f}(\xi)\overline{\tilde{\chi}(g^{-1}\xi)} = 0,$$

where \tilde{f} and $\tilde{\chi}$ are the Fourier transforms of f and χ , respectively, and the overbar stands for the complex conjugate. The Fourier transform of f is understood in the sense of distributions. The Fourier transform $\tilde{\chi}$ is an entire function of exponential type because function χ has support \overline{D} , which is a bounded set. Moreover, $\tilde{\chi}$ is a uniformly bounded function of $\xi \in \mathbb{R}^n$. The product of the tempered distribution \tilde{f} and the function $\tilde{\chi}$ is a tempered distribution also, that is, an element of \mathcal{S}' .

Since g^{-1} runs through all the rotations, one can replace g^{-1} by g . It follows from (21) that

$$(22) \quad \text{supp } \tilde{f} = \cup_k C_k, \text{ where } C_k := \{\xi : \tilde{\chi}(\xi) = 0 \quad \forall \xi : \xi^2 - k^2 = 0\}.$$

In other words, the support of the distribution \tilde{f} is a subset of the union of spherical surfaces of zeros of $\tilde{\chi}$, the Fourier transform of the characteristic function of the bounded domain D . Since $\tilde{\chi}(\xi)$ is an entire function of exponential type, vanishing on an irreducible algebraic variety $\xi^2 - k^2 = 0$ in \mathbb{C}^3 , one concludes, using the division lemma, that

$$(23) \quad \tilde{\chi}(\xi) = (\xi^2 - k^2)\tilde{u}(\xi),$$

where \tilde{u} is an entire function of the same exponential type as $\tilde{\chi}$ (see [3]). Therefore, by the Paley-Wiener theorem, the corresponding u has compact support. Taking the inverse Fourier transform of equation (23), one gets:

$$(24) \quad (-\nabla^2 - k^2)u(x) = \chi(x) \quad \text{in } \mathbb{R}^3, \quad u = 0 \quad \text{if } |x| > R,$$

where $R > 0$ is sufficiently large. By the elliptic regularity results, one concludes that $u \in H_{loc}^2(\mathbb{R}^3)$. Since u solves the Helmholtz elliptic equation and vanishes near infinity, that is, in the region $|x| > R$, the uniqueness of the solution to the Cauchy problem to the equation (24) and the path connectedness of the complement $D_1 := \bar{D}'$ of the closure \bar{D} of D allow one to conclude that $u = 0$ in D_1 . The connectedness and path connectedness of D_1 follow from our Assumptions A and from the isotopy extension theorem (see [4]). If $u = 0$ in D_1 and $u \in H_{loc}^2(\mathbb{R}^3)$, it follows from the Sobolev embedding theorem that the boundary conditions (2) hold. Since $\chi(x) = 1$ in D , equation (2) holds. The converse statement in Theorem 3 has already been established above, in our discussion of the equivalence of Conjectures 1 and 2. Theorem 3 is proved. \square .

3 Relation to analyticity

The classical Morera theorem in complex analysis says that if $\int_C f(z)dz = 0$ for any closed polygon C in a domain D of the complex plane, and if f is continuously differentiable in D , then f is analytic in D . A simple proof is based on a version of Green's formula:

$$0 = \int_C f(z)dz = 2i \int_{\Delta} \bar{\partial} f dx dy.$$

Here Δ is the plane domain with the boundary C and $\bar{\partial} f := \frac{f_x + if_y}{2}$. If $\int_{\Delta} \bar{\partial} f dx dy = 0$ for any polygon Δ , then one passes to the limit in the formula

$$\frac{1}{|\Delta|} \int_{\Delta} \bar{\partial} f dx dy = 0,$$

where $|\Delta|$ is the area of Δ and the limit is taken as $\text{diam} \Delta \rightarrow 0$, so that Δ shrinks uniformly in directions to a point $(x, y) \in \Delta$. Then for almost all points in D one gets $\bar{\partial} f = 0$, and if $\bar{\partial} f$ is continuous, then $\bar{\partial} f = 0$ everywhere in D . This implies that f is analytic in D .

One may ask if the assumption that f is continuously differentiable can be replaced by a weaker assumption, and if the set of polygons can be replaced by some other sets. The answer to the first question is easy: if $f \in L^1_{loc}(D)$, then one considers a mollified function $f_{\epsilon}(z) := \int_{\zeta: |z-\zeta| \leq \epsilon} \omega_{\epsilon}(z - \zeta) f(\zeta) d\zeta$, where $\zeta = u + iv$ and $\omega_{\epsilon}(z)$ is the standard mollifying kernel ([5], p.14). It is known that $f_{\epsilon} \rightarrow f$ in $L^1(D)$ as $\epsilon \rightarrow 0$, and one can select a subsequence $\epsilon_j \rightarrow 0$, such that $f_{\epsilon_j} \rightarrow f$ almost everywhere in D . If $\int_C f(z)dz = 0$ for any closed polygon C , then $\int_C f_{\epsilon}(z)dz = 0$ for any closed polygon C , and the above argument, applied to the C^1 -smooth f_{ϵ} , leads to the conclusion that f_{ϵ} is analytic in D for all sufficiently small ϵ . Since a sequence f_{ϵ_j} of analytic functions converges to f in $L^1(D)$ and almost everywhere in D , one concludes that f is analytic in D . This follows from the closedness of the differential operator $\bar{\partial}$. Namely, one has $\|f_{\epsilon_j} - f\|_{L^1(D)} \rightarrow 0$ and $\bar{\partial} f_{\epsilon_j} = 0$, so $\|\bar{\partial} f_{\epsilon_j} - 0\|_{L^1(D)} \rightarrow 0$. Consequently, f belongs to the domain of the operator $\bar{\partial}$, and $\bar{\partial} f = 0$ in $L^1(D)$. Therefore, f is analytic in D .

The second question: can one replace the set of polygons by other sets is less simple. For example, one cannot replace polygons by the set $\sigma(B)$, where B is a ball. Indeed, using the above argument one arrives at the relation $\int_{\sigma(B)} \bar{\partial} f dx dy = 0$, and this does not imply that $\bar{\partial} f = 0$, as the example on p. 2 shows. However, any domain D which has P -property can be used in a generalization of the Morera theorem. By ∂D the boundary of D is denoted.

Theorem 4. Assume that D has P -property, $f \in L^1_{loc}(\mathbb{R}^2)$, and

$$\int_{\partial\sigma(D)} f dz = 0 \quad \forall \sigma \in G.$$

Then f is an entire function.

Proof. By the argument given above it is sufficient to prove this theorem assuming f C^1 -smooth. In this case one has

$$(25) \quad \int_{\partial\sigma(D)} f dz = 2i \int_{\sigma(D)} \bar{\partial} f dx dy = 0,$$

where $z = x + iy$. Since D has P -property, one concludes from the above equation that $\bar{\partial} f = 0$ in \mathbb{R}^2 . This means that f is an entire function. \square

Let B_r denote a ball (disc if $n = 2$) of radius r centered at the origin, $f \in L^1_{loc}(\mathbb{R}^2)$, and s_j , $j = 1, 2, \dots$, denote positive zeros of the Bessel function $J_1(s)$.

From here to the end of Section 3 it is assumed that $n = 2$.

In [16] the following result is proved:

If $\int_{\partial\sigma(B_r)} f dz = 0$ for $r = r_1$ and for $r = r_2$, and if r_1/r_2 does not belong to the set s_j/s_m for any positive integers j and m , then f is an entire function.

This result is an immediate consequence of Theorem 4. Indeed, it follows from Example 1 with $n = 2$ that if (1) holds for $D = B_{r_1}$, then the support of f belongs to the set $\mathcal{N}^{(1)} = \{\xi : |\xi| = s_j/r_1, \xi \in \mathbb{R}^2 \text{ for some positive integers } j\}$, see the proof of Theorem 3.

Similarly, if (1) holds for $D = B_{r_2}$, then the support of f belongs to the set $\mathcal{N}^{(2)} = \{\xi : |\xi| = s_m/r_2, \xi \in \mathbb{R}^2 \text{ for some positive integers } m\}$. If these two sets, $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$, have empty intersection, then the support of f is empty, so that $f = 0$. The role of f will be played by $\bar{\partial} f$ in what follows.

The set $\mathcal{N}^{(1)}$ does not intersect the set $\mathcal{N}^{(2)}$ if and only if $s_j/r_1 \neq s_m/r_2$ for any positive integers j and m . This condition is equivalent to the condition that r_1/r_2 does not belong to the set s_j/s_m for any positive integers j and m . Under this condition one concludes that $\bar{\partial} f = 0$, because the role of f is played in our case by $\bar{\partial} f$, see equation (25). Thus, f is an entire function. \square

The proof of this result in [16] is much longer and more complicated.

4 Another approach to Pompeiu problem and some remarks

In Conjecture 3 we assume that

$$(26) \quad \int_D e^{ik\alpha \cdot x} dx = 0 \quad \forall \alpha \in S^2, \quad k = \text{const} > 0.$$

In the derivation of this equation α can be an arbitrary complex vector $z \in M$, where $M \subset \mathbb{C}^3$ is an algebraic variety defined by the equation $z \cdot z = 1$. Here $z \in \mathbb{C}^3$ and $z \cdot w := \sum_{j=1}^3 z_j w_j$. Note that w_j is used in the definition of $z \cdot w$, and not its complex conjugate \bar{w}_j . Let $z = a + ib$, where $a, b \in \mathbb{R}^3$. One checks easily that $z = a + ib \in M$ if and only if $a \cdot b = 0$ and $a^2 - b^2 = 1$, where $a^2 := a \cdot a$. Let $a = (\lambda^2 + 1)^{1/2}(e_1 \cos \theta + e_2 \sin \theta)$, where e_j , $j = 1, 2, 3$, are unit vectors of a Cartesian basis in \mathbb{R}^3 , and $\theta \in [0, 2\pi)$. Let $b = \lambda e_3$. Here $\lambda \in \mathbb{R}$ is an arbitrary number. One can easily check that $\pm a \pm ib \in M$.

Equation (26) becomes

$$(27) \quad \int_D e^{\pm \lambda k x_3 + ik(\lambda^2 + 1)^{1/2}(x_1 \cos \theta + x_2 \sin \theta)} dx_1 dx_2 dx_3 = 0.$$

Assume that $D \subset \mathbb{R}^3$ is a bounded domain diffeomorphic to a ball.

Conjecture 6. *Under these assumptions equation (27) holds for all $\lambda \in \mathbb{R}^1$ and all $\theta \in [0, 2\pi)$ if and only if D is a ball.*

In order to prove Conjecture 6 it may help to assume additionally that D is a convex centrally symmetric domain, but the author thinks that Conjecture 6 is correct without additional assumptions.

If D is a ball of radius R , then equation (27) holds if $kR = s_{\frac{3}{2},j}$, where $s_{\frac{3}{2},j}$ are positive zeros of the Bessel function $J_{3/2}$. This follows from the calculations given in Example 1. Conjecture 6, if it is correct, says that equation (27) cannot hold for any domain, satisfying the stated assumptions, except for a ball. Asymptotic behavior, as $\lambda \rightarrow \infty$, of the integral in (27) may help to verify Conjecture 6.

Remark 2. It is proved in [8] and in [10] that if $D_1 \in P$ and D_2 is a "sufficiently close" to D_1 , then $D_2 \in P$. This means that P -property is *stable* in some sense. The \bar{P} -property is *not stable*: small perturbations of D lead to domains the Fourier transform of the characteristic function of which do not have a spherical surface of zeros.

The notion of being "sufficiently close" is defined as follows. The domain D_2 is C^3 -smooth, strictly convex, its Gaussian curvature is bounded from below by a positive constant, and $\text{meas}(D_{12} \setminus D^{12})$ is sufficiently small. Here $D^{12} := D_1 \cap D_2$ and $D_{12} := D_1 \cup D_2$.

Remark 3. One can prove (see, for example, [10], p.412) that if $D \subset \mathbb{R}^n$ is a bounded strictly convex domain with a smooth boundary, and $\tilde{\chi}_D(t_m \alpha) = 0$ for all $\alpha \in S^{n-1}$ and a sequence $t_m \rightarrow +\infty$, then D is a ball.

Remark 4. It is easy to give examples of the domains D which have P -property: any polygon has this property since it does not have a real-analytic boundary. An ellipsoid in \mathbb{R}^n , $n \geq 2$, has P -property unless it is a ball. This is easy to check by calculating the Fourier transform of the characteristic function of an ellipsoid and checking that this Fourier transform does not have a spherical set of zeros. If the equation of the ellipsoid D is $\sum_{j=1}^n \frac{x_j^2}{a_j^2} = 1$, then this Fourier transform is

$$\tilde{\chi}_D = (2\pi)^{n/2} \left(\sum_{j=1}^n \xi_j^2 a_j^2 \right)^{-\frac{n}{2}} J_{n/2} \left(\left(\sum_{j=1}^n \xi_j^2 a_j^2 \right)^{1/2} \right).$$

This is calculated by making the change of variables $x'_j = x_j/a_j$, which transforms the ellipsoid into the ball of radius 1 and the ξ_j variable of the Fourier transform into $\xi_j a_j$, so that then the formula for the Fourier transform of the characteristic function of the ball can be used. This formula is given in Example 1.

Remark 5. One can construct $f \neq 0$ satisfying equation (1) for the domain D , which fails to have P -property, by the following method, see [10], p.406. Let $|\xi| = b$ be the spherical surface S_b of zeros of the Fourier transform of the characteristic function of D . Take any function $A(\xi) \in L^1(S_b)$ and define $\tilde{f}(\xi) = A(\xi)\delta(|\xi| - b)$, where $\delta(|\xi| - b)$ is the delta-function supported on the sphere S_b . Then the inverse Fourier transform of \tilde{f} is a function $f \neq 0$, which satisfies equation (1). Since \tilde{f} has compact support, by the Paley-Wiener theorem the function f is an entire function of x . For example, if $n = 3$ and $A(\xi) = 1$, then

$$f(x) = \frac{1}{(2\pi)^3} \int_{|\xi|=b} e^{-i\xi \cdot x} d\xi = (2\pi)^{-\frac{3}{2}} b^2 \frac{J_{1/2}(b|x|)}{\sqrt{b|x|}},$$

where the known formula

$$e^{ik\alpha \cdot x} = \sum_{\ell=0}^{\infty} 4\pi i^\ell j_\ell(k|x|) Y_\ell(\alpha) Y_\ell(x^0)$$

was used. In this formula $k > 0$ is a constant, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in \mathbb{R}^3 , Y_ℓ are the normalized in $L^2(S^2)$ spherical harmonics, $x^0 := x/|x|$, and $j_\ell(r) := \sqrt{\frac{\pi}{2r}} J_{\ell+\frac{1}{2}}(r)$.

Acknowledgments. The author thanks Professors R. Burckel and C.N. Moore for reading the paper and comments.

References

- [1] L. Brown, B. Schreiber, B. Taylor, Spectral synthesis and the Pompeiu problem, *Ann. Inst. Fourier* **23** no. 3 (1973) 125-154.
- [2] L. Chakalov, Sur un problème de D.Pompeiu, *Godishnik Univ. Sofia, Fac. Phys-Math.* **40** (1944) 1-14.
- [3] B. Fuks, *Theory of analytic functions of several variables*. AMS, Providence RI, 1963.
- [4] M. Hirsch, *Differential topology*. Springer-Verlag, New York, 1976.
- [5] L.Hörmander, The analysis of linear partial differential operators I. Springer-Verlag, Berlin, 1983.
- [6] T. Kobayashi, Asymptotic behavior of the null variety for a convex domain in a non-positively curved space form, *J. Fac. Sci.Univ. Tokyo Sect IA Math.***36** (1989) 389-478.
- [7] D. Pompeiu, Sur une propriété intégrale des fonctions de deux variables reelles, *Bull. Sci. Acad. Roy. Belgique* **5** no 15 (1929) 265-269.
- [8] A. G. Ramm, The Pompeiu problem, *Applicable Analysis* **64** no 1-2 (1997) 19-26.
- [9] A. G. Ramm, Necessary and sufficient condition for a domain, which fails to have Pompeiu property, to be a ball, *Journ. of Inverse and Ill-Posed Probl.* **6** no 2, (1998), 165-171.
- [10] A. G. Ramm, *Inverse Problems*. Springer, New York, 2005.
- [11] A. G. Ramm, A symmetry problem, *Ann. Polon. Math.* **92** (2007) 49-54.
- [12] A. G. Ramm, Symmetry problems 2, *Annal. Polon. Math.* **96** no 1 (2009) 61-64.
- [13] A. G. Ramm, Symmetry problem, *Proc. Amer. Math. Soc.* (forthcoming)
- [14] S. Williams, Analyticity of the boundary for Lipschitz domains without Pompeiu property, *Indiana Univ. Math. Journ.* **30** (1981) 357-369.
- [15] L. Zalcman, A bibliographical survey of the Pompeiu Problem, in the book *Approximation by solutions of partial differential equations*. Edited by B.Fuglede. Kluwer Acad., Dordrecht, 1992, pp. 177-186.

- [16] L. Zalcman, Analyticity and the Pompeiu problem, *Arch. Rat. Mech. Anal.* **47** (1972) 237-254.

Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602,
email: ramm@math.ksu.edu